

EQUIVARIANT K -THEORY OF GRASSMANNIANS II: THE KNUTSON-VAKIL CONJECTURE

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ABSTRACT. In 2005, A. Knutson–R. Vakil conjectured a *puzzle* rule for equivariant K -theory of Grassmannians. We resolve this conjecture. After giving a correction, we establish a modified rule by combinatorially connecting it to the authors’ recently proved tableau rule for the same Schubert calculus problem.

CONTENTS

1. Introduction	1
1.1. The puzzle conjecture	2
1.2. A modified puzzle rule	3
2. A tableau rule for $K_{\lambda, \mu}^\nu$	3
3. Proof of Theorem 1.2: bijecting the tableau and puzzle rules	5
3.1. Description of the bijection	5
3.2. Well-definedness of $\phi : \mathcal{P} \rightarrow \mathcal{T}$	8
3.3. Well-definedness of $\psi : \mathcal{T} \rightarrow \mathcal{P}$	9
3.4. Weight-preservation	10
Acknowledgments	10
References	10

1. INTRODUCTION

A. Knutson–R. Vakil [CoVa05, §5] conjectured a combinatorial rule for the structure coefficients of the torus-equivariant K -theory ring of a Grassmannian. The structure coefficients are with respect to the basis of Schubert structure sheaves. Their rule extends *puzzles*, combinatorial objects founded in work of A. Knutson–T. Tao [KnTa03] and in their collaboration with C. Woodward [KnTaWo04]. The various puzzle rules play a prominent role in modern Schubert calculus, see e.g., [BuKrTa03, Va06, CoVa05], recent developments [Kn10, KnPu11, BKPT13, Bu15] and the references therein.

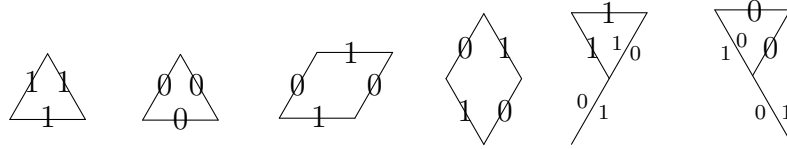
This paper is a sequel to [PeYo15] where we gave the first proved tableau rules for these structure coefficients, including a conjecture of H. Thomas and the second author [ThYo13]. Here we use these results to prove a mild correction of the puzzle conjecture.

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1.1. The puzzle conjecture. Let $X = \text{Gr}_k(\mathbb{C}^n)$ denote the Grassmannian of k -dimensional subspaces of \mathbb{C}^n . The general linear group GL_n acts transitively on X by change of basis. The Borel subgroup $B \subset \text{GL}_n$ of invertible lower triangular matrices acts on X with finitely many orbits, i.e., the **Schubert cells** X_λ° . These orbits are indexed by $\{0, 1\}$ -sequences λ of length n with k -many 1's. The **Schubert varieties** are the Zariski closures $X_\lambda := \overline{X_\lambda^\circ}$. The X_λ are stable under the action of the maximal torus $T \subset B$ of invertible diagonal matrices. Therefore their structure sheaves \mathcal{O}_{X_λ} admit classes in $K_T(X)$, the Grothendieck ring of T -equivariant vector bundles over X . Now, $K_T(X)$ is a $K_T(\text{pt})$ -module and the $\binom{n}{k}$ Schubert classes form a module basis. One may make a standard identification $K_T(\text{pt}) \cong \mathbb{Z}[1 - \frac{t_i}{t_{i+1}} : 1 \leq i < n]$. The structure coefficients $K_{\lambda, \mu}^\nu \in K_T(\text{pt})$ are defined by

$$[\mathcal{O}_{X_\lambda}] \cdot [\mathcal{O}_{X_\mu}] = \sum_{\nu} K_{\lambda, \mu}^\nu [\mathcal{O}_{X_\nu}].$$

Consider the n -length equilateral triangle oriented as Δ . A **puzzle** is a filling of Δ with the following **puzzle pieces**:



The double-labeled edges are **gashed**. A **filling** requires that the common (non-gashed) edges of adjacent puzzle pieces share the same label. Two gashed edges may not be overlayed. The pieces on either side of a gash must have the indicated labels. The first three may be rotated but the fourth (**equivariant piece**) may not [KnTa03]. We call the remainder **KV-pieces**; these may not be rotated. The fifth piece may only be placed if the equivariant piece is attached to its left. There is a “nonlocal” requirement [CoVa05, §5] for using the sixth piece: it “may only be placed (when completing the puzzle from top to bottom and left to right as usual) if the edges to its right are a (possibly empty) series of horizontal 0's followed by a 1.” A **KV-puzzle** is a puzzle filling of Δ .

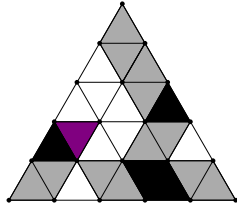
Let $\Delta_{\lambda, \mu, \nu}$ be Δ with the boundary given by

- λ as read \nearrow along the left side;
- μ as read \searrow along the right side; and
- ν as read \rightarrow along the bottom side.

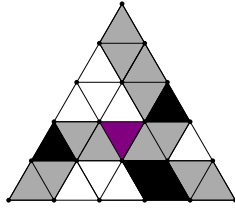
The **weight** $\text{wt}(P)$ of a **KV-puzzle** P is a product of the following factors. Each KV-piece contributes a factor of -1 . For each equivariant piece one draws a \searrow diagonal arrow from the center of the piece to the ν -side of Δ ; let a be the unit segment of the ν -boundary, as counted from the right. Similarly one determines b by drawing a \swarrow antidiagonal arrow. The equivariant piece contributes a factor of $1 - \frac{t_a}{t_b}$.

Conjecture 1.1 (The Knutson-Vakil puzzle conjecture). $K_{\lambda, \mu}^\nu = \sum_P \text{wt}(P)$ where the sum is over all KV-puzzles of $\Delta_{\lambda, \mu, \nu}$.

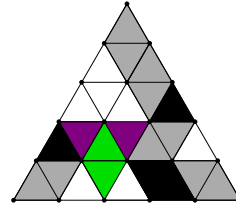
We consider the structure coefficient $K_{01001, 00101}^{10010}$ for $\text{Gr}_2(\mathbb{C}^5)$. The reader can check that there are six KV-puzzles P_1, P_2, \dots, P_6 with the indicated weights. Henceforth, we color-code the six puzzle pieces black, white, grey, green, yellow and purple, respectively.



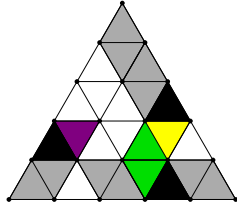
$$\text{wt}(P_1) = -1$$



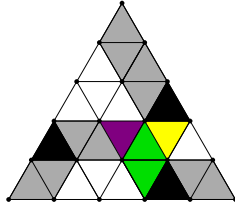
$$\text{wt}(P_2) = -1$$



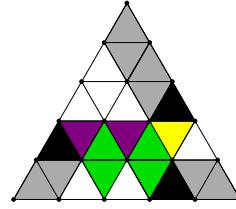
$$\text{wt}(P_3) = (-1)^2(1 - \frac{t_3}{t_4})$$



$$\text{wt}(P_4) = (-1)^2(1 - \frac{t_2}{t_3})$$





$$\text{wt}(P_5) = (-1)^2(1 - \frac{t_2}{t_3})$$



$$\text{wt}(P_6) = (-1)^3(1 - \frac{t_3}{t_4})(1 - \frac{t_2}{t_3})$$

Using double Grothendieck polynomials [LaSc82] (see also [FuLa94] and references therein), one computes $K_{01001,00101}^{10010} = -(1 - \frac{t_2}{t_4}) = \text{wt}(P_2) + \text{wt}(P_3) + \text{wt}(P_5) + \text{wt}(P_6)$. This gives a counterexample to Conjecture 1.1. Actually, this subset of four puzzles witnesses the rule of Theorem 1.2 below.

1.2. A modified puzzle rule. We define a **modified KV-puzzle** to be a KV-puzzle with the nonlocal condition on the second KV-piece replaced by the requirement that the second KV-piece only appears in the combination pieces  or .

Theorem 1.2. $K_{\lambda,\mu}^\nu = \sum_P \text{wt}(P)$ where the sum is over all modified KV-puzzles of $\Delta_{\lambda,\mu,\nu}$.

We have a few remarks. First, the rule of Theorem 1.2 is “positive” in the sense of D. Anderson-S. Griffeth-E. Miller’s [AnGrMi11]; cf. the discussion in [PeYo15, §1.4]. Second, it is a natural objective to interpret Theorem 1.2 via geometric degeneration; see [CoVa05, Kn10]. Third, the first author has found a tableau formulation similar to that of [PeYo15] to complement the puzzle rule of [Kn10] for the *different* Schubert calculus problem in $K_+(X)$ of multiplying a class of a Schubert variety by that of an opposite Schubert variety; further discussion may appear elsewhere.

To prove Theorem 1.2, we first give a variant of the main theorem of [PeYo15]; see Section 2. In Section 3, we then give a weight-preserving bijection between modified KV-puzzles and the objects of the rule of Section 2.

2. A TABLEAU RULE FOR $K_{\lambda,\mu}^\nu$

We need to briefly recall the definitions of [PeYo15, § 1.2–1.3]; there the Schubert varieties X_λ are indexed by Young diagrams λ contained in a $k \times (n - k)$ rectangle. (Throughout, we orient Young diagrams and tableaux according to the English convention.)

An **edge-labeled genomic tableau** is a filling of the boxes and horizontal edges of a skew diagram ν/λ with subscripted labels i_j , where i is a positive integer and the j ’s that appear for each i form an initial interval of positive integers. Each box of ν/λ contains

one label, whereas the horizontal edges weakly between the southern border of λ and the northern border of ν are filled by (possibly empty) sets of labels. A genomic edge-labeled tableau T is **semistandard** if

- (S.1) the box labels of each row strictly increase lexicographically from left to right;
- (S.2) ignoring subscripts, each label is strictly less than any label strictly south in its column;
- (S.3) ignoring subscripts, the labels appearing on a given edge are distinct;
- (S.4) if i_j appears strictly west of i_k , then $j \leq k$.

Index the rows of ν from the top starting at 1. We say a label i_j is **too high** if it appears weakly above the north edge of row i . We refer to the collection of all i_j 's (for fixed i, j) as a **gene**. The **content** of T is the composition $(\alpha_1, \alpha_2, \dots)$ where α_i is greatest so that i_{α_i} is a gene of T .

Recall that in the classical tableau theory, a semistandard tableau S is *ballot* if, reading the labels down columns from right to left, we obtain a word W with the following property: For each i , every initial segment of W contains at least as many i 's as $(i + 1)$'s. Given an edge-labeled genomic tableau T , choose one label from each gene and delete all others; now delete all subscripts. We say T is **ballot** if, regardless of our choices from genes, the resulting tableau (possibly containing holes) is necessarily ballot in the above classical sense. (In the case of multiple labels on a edge, read them from least to greatest.)

We now diverge slightly from the treatment of [PeYo15], borrowing notation from [ThYo13]. Given a box x in an edge-labeled genomic tableau T , we say x is **starrable** if it contains i_j , is in row $> i$, and i_{j+1} is not a box label to its immediate right. Let $\text{StarBallotGen}_\mu(\nu/\lambda)$ be the set of all ballot semistandard edge-labeled genomic tableaux of shape ν/λ and content μ with no label too high, where the label of each starrable box may freely be marked by \star or not. The tableau T illustrated in Figure 2 is an element of $\text{StarBallotGen}_{(10,5,3)}((15, 8, 5)/(12, 2, 1))$. There are three starrable boxes in T , in only one of which the label has been starred.

Let $\text{Man}(x)$ denote the length of any $\{\uparrow, \rightarrow\}$ -lattice path from the southwest corner of $k \times (n - k)$ to the northwest corner of x . For x in row r containing i_j^* , set $\text{starfactor}(x) := 1 - \frac{t_{\text{Man}(x)+1}}{t_{r-i+\mu_i-j+1+\text{Man}(x)}}$. For an edge label $\ell = i_j$ in the southern edge of x in row r , set $\text{edgefactor} := 1 - \frac{t_{\text{Man}(x)}}{t_{r-i+\mu_i-j+1+\text{Man}(x)}}$. Finally for $T \in \text{StarBallotGen}_\mu(\nu/\lambda)$, define

$$\widehat{\text{wt}}(T) := (-1)^{\hat{d}(T)} \times \prod_{\ell} \text{edgefactor}(\ell) \times \prod_x \text{starfactor}(x);$$

here the products are respectively over edge labels ℓ and boxes x containing starred labels, while $\hat{d}(T) := \#(\text{labels in } T) + \#(\star\text{'s in } T) - |\mu|$. Let

$$\hat{L}_{\lambda, \mu}^\nu := \sum_T \widehat{\text{wt}}(T),$$

where the sum is over all $T \in \text{StarBallotGen}_\mu(\nu/\lambda)$.

We need a reformulation of [PeYo15, Theorem 1.3]; the proof is a simple application of the “inclusion-exclusion” identity $\prod_{i \in [m]} a_i = \sum_{S \subseteq [m]} (-1)^{|S|} \prod_{i \in S} (1 - a_i)$.

Theorem 2.1. $K_{\lambda, \mu}^\nu = \hat{L}_{\lambda, \mu}^\nu$. □

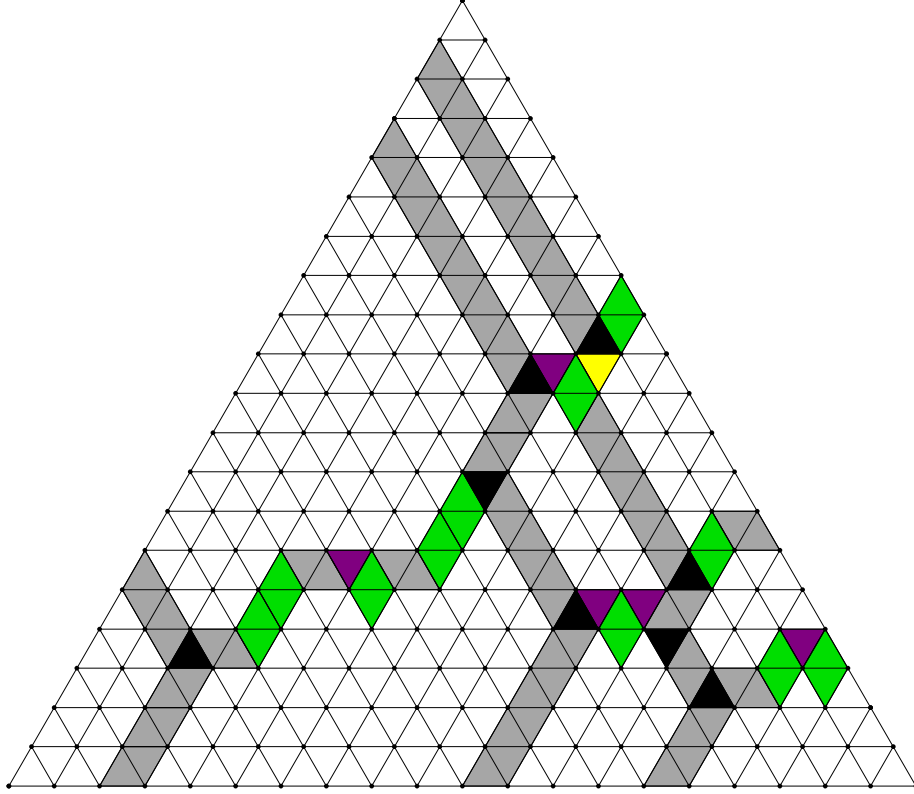


FIGURE 1. A “generic” modified KV-puzzle P ($k = 3, n = 20$).

Example 2.2. Let $k = 2, n = 5$ and $\lambda = (2, 0), \mu = (1, 0)$ and $\nu = (3, 1)$. The four tableaux contributing to $\hat{L}_{\lambda, \mu}^{\nu}$ are

		1_1
1_1		

		1_1
1_1	1_1	

		1_1
1_1^*		

		1_1
1_1^*	1_1	

$$\widehat{\text{wt}}(T_2) = -1 \quad \widehat{\text{wt}}(T_3) = (-1)^2(1 - \frac{t_3}{t_4}) \quad \widehat{\text{wt}}(T_5) = (-1)^2(1 - \frac{t_2}{t_3}) \quad \widehat{\text{wt}}(T_6) = (-1)^3(1 - \frac{t_3}{t_4})(1 - \frac{t_2}{t_3})$$












Our indexing of these tableaux alludes to the precise connection to the four puzzles P_2, P_3, P_5 and P_6 of Section 1.1, as explained in the next section. \square

3. PROOF OF THEOREM 1.2: BIJECTING THE TABLEAU AND PUZZLE RULES

3.1. Description of the bijection. To relate the modified KV-puzzle rule of Theorem 1.2 with the tableau rule of Theorem 2.1, we give a variant of T. Tao’s “proof without words” [Va06] (and its modification by K. Purbhoo [Pu08]) that bijects cohomological puzzles (using the first three pieces) and a tableau Littlewood-Richardson rule. An extension of this proof for equivariant puzzles (i.e., fillings that additionally use the equivariant piece) was given by V. Kreiman [Kr10]; we also incorporate elements of his bijection in our analysis.

Figure 1 gives a “generic” example of a (modified) KV-puzzle P . We will define a **track** π_i from the i th 1 (from the left) on the ν -boundary of $\Delta_{\lambda, \mu, \nu}$ to the i th 1 (from the top) on

the μ -boundary. To do this, we describe the **flow** through the (oriented, non-KV) puzzle pieces that use a 1 and four **combination pieces** (possible ways one can use the KV-pieces under the rules for a modified KV-puzzle):

- (A.1)  : go northeast
- (A.2)  : go north then northeast
- (A.3)  : go left to right
- (A.4)  : go northeast
- (A.5)  : go in through the north \ of the purple triangle, come out northeast from the purple gash into the southwest \ of the green rhombus and pass northeast through this rhombus
- (A.6)  : come in through the left side and out the top
- (A.7)  : come in through the southwest side of the green rhombus and out the top of the yellow triangle
- (A.8)  : come in through the north \ of the purple triangle, out the gash into the \ of the  , out the — of  into the bottom of the grey rhombus and out its top
- (A.9)  : come into the north \ of the purple triangle, out the gash into the southwest \ of the green rhombus and out the northeast \ into the left side of the yellow triangle and then go out the — of that triangle.

Thinking of the (combination) pieces in (A.1)–(A.9) as letters of an alphabet, we can encode the northmost track in P (from Figure 1) as the word

$$\text{grey rhombus}^3 \text{ black triangle} \text{ grey rhombus} \text{ green rhombus}^2 \text{ grey rhombus} \text{ purple triangle} \text{ grey rhombus} \text{ green rhombus}^2 \text{ black triangle pointing down} \text{ grey rhombus}^2 \text{ black triangle} \text{ yellow triangle} \text{ black triangle} \text{ green rhombus}.$$

Recall, if κ is a letter/word in some alphabet, then the **Kleene star** is $\kappa^* := \{\emptyset, \kappa, \kappa\kappa, \dots\}$.




Proposition 3.1 (Decomposition of π_i). *The list of (combination) pieces that appear in π_i , as read from southwest to northeast, is a word from the following formal grammar:*


$$(3.1) \quad \text{boxes}[\text{edges startrow boxes}]^* \text{edges}$$



where

$$\begin{aligned} \text{boxes} &:= \text{grey rhombus}^* \text{ black triangle} \\ \text{edges} &:= [\text{grey rhombus}^* \text{ green rhombus}^* \text{ grey rhombus}^* \text{ purple triangle}^*]^* \\ \text{startrow} &:= \text{black triangle pointing down} \cup \text{yellow triangle} \cup \text{black triangle pointing down} \cup \text{purple triangle} \end{aligned}$$

Proof. By inspection of the rules for modified KV-puzzles. □

The remaining filling of the puzzle is forced, which we explain in two steps. First there is the **NWray** of each , i.e., the (possibly empty) path of upward pointing grey rhombi  growing from the / of this .










Lemma 3.2. *The NWray of  ends either at the λ -boundary of Δ or with a piece from startrow. In the latter case, the shared edge is the south-then-eastmost edge of the (combination) piece.*

Proof. The north / of  is labeled 1. By inspection, the only (combination) pieces that can connect to this edge are  and those from startrow (at the stated shared edge). □

Second, pieces of the puzzle not in a track or NWray are 0-triangles (depicted white).

We correspond Young diagrams to $\{0, 1\}$ -sequences. Trace the $\{\leftarrow, \downarrow\}$ -lattice path defined by the southern boundary of λ (as placed in the northwest corner of $k \times (n - k)$) starting from the northeast corner of $k \times (n - k)$ towards the southeast corner of $k \times (n - k)$. Record each \leftarrow step with “0” and each \downarrow step with “1”.

We now convert P into (we claim) an edge-labeled starred genomic tableau $T := \phi(P)$ of shape ν/λ with content μ . The placement of the labels of family i is governed by the decomposition (3.1) of π_i . The initial sequence of k \blacktriangleleft 's indicates the leftmost possible placement of box labels $i_{\mu_i}, i_{\mu_i-1}, \dots, i_{\mu_i-k+1}$ (from right to left) in row i of T . Continuing to read the sequence, one interprets:

- (B.1)  \leftrightarrow "place (unstarred) box label of next smaller gene"
 (B.2)  \leftrightarrow "end placing box labels in current row"
 (B.3)  \leftrightarrow "skip to the next column left"
 (B.4)  \leftrightarrow "place lower edge label of the next smaller gene"
 (B.5)  \leftrightarrow "place lower edge label of the same gene last used"
 (B.6)  \leftrightarrow "go to next row"
 (B.7)  \leftrightarrow "go to next row and place \star -ed box label of the next smaller gene"
 (B.8)  \leftrightarrow "go to next row and place (unstarred) box label of the same gene last used"
 (B.9)  \leftrightarrow "go to next row and place \star -ed box label of the same gene last used".

Applying ϕ to the puzzle P of Figure 1 gives the tableau T of Figure 2. Here, $\lambda = 0^5 10^{10} 1010$, corresponding to the inner shape $(12, 2, 1)$ (which is shaded in grey). Since $\mu = 0^7 10^5 10^2 10^3$, the content of T is $(10, 5, 3)$. Finally, since $\nu = 0^2 10^7 10^3 10^5$, the outer shape of T is $(15, 8, 5)$. As another example, ϕ connects the puzzles P_2, P_3, P_5 and P_6 of Section 1 respectively with the tableaux T_2, T_3, T_5 and T_6 of Example 2.2.

[illegible]

FIGURE 2. The tableau $T := \phi(P)$ corresponding to the modified KV-puzzle P of Figure 1.

Conversely, given $T \in \text{StarBallotGen}_\mu(\nu/\lambda)$, construct a word σ_i using the correspondences (B.1)–(B.9), for $1 \leq i \leq k$. That is, read the occurrences (possibly zero) of family i in T from right to left and from the i th row down. (Note about (B.6) in the degenerate case that there are no labels of family i in the next row: use \blacktriangle after reading the leftmost column in ν/λ that does not have any labels of family $< i$.)

Lemma 3.3. *Each σ_i is of the form (3.1).*

Proof. Since T is semistandard, in any row, all box labels of family i are contiguous and strictly right of any (lower) edge labels of that family on that row. The lemma follows. \square

We describe a claimed filling $P := \psi(T)$ of $\Delta_{\lambda,\mu,\nu}$. There are k 1's on each side of $\Delta_{\lambda,\mu,\nu}$; to the i th 1 from the left on the ν -boundary of $\Delta_{\lambda,\mu,\nu}$, place puzzle pieces in the

order indicated by σ_i . That is attach the next (combination) piece using the northmost \setminus edge on its west side, if it exists. Otherwise attach at the piece's unique southern edge. We attach at the unique \rightarrow or \setminus edge of the thus far constructed track. Fill in the order $i = 1, 2, 3, \dots, k$. Now stack \blacklozenge 's northwest of each \blacktriangle until (we claim) it reaches one of the pieces of (A.6)–(A.9) at the southmost $/$ edge, or the λ -boundary of $\Delta_{\lambda, \mu, \nu}$. Complete using white triangles.

Sections 3.2–3.4 prove ϕ and ψ are well-defined and weight-preserving maps between

$$\mathcal{P} := \{\text{modified KV-puzzles of } \Delta_{\lambda, \mu, \nu}\} \text{ and } \mathcal{T} := \text{StarBallotGen}_\mu(\nu/\lambda).$$

Semistandardness (specifically (S.4)) implies that knowing the locations of labels of family i , and which labels are repeated or \star -ed, uniquely determines the gene(s) in each location. The injectivity of ϕ and ψ is easy from this. Moreover, by construction (cf. Lemma 3.3), the two maps are mutually reversing. Thus, Theorem 1.2 follows from Theorem 2.1. \square

3.2. Well-definedness of $\phi : \mathcal{P} \rightarrow \mathcal{T}$. Let $P \in \mathcal{P}$ be a modified KV-puzzle for $\Delta_{\lambda, \mu, \nu}$. For the track π_i , let $\blacktriangle_{i,j}$ refer to the j th black triangle seen along π_i (as read from southwest to northeast). Let \mathbb{S} denote any of the (combination) pieces that appear in startrow. Similarly, we let $\mathbb{S}_{i,j}$ be the j th such piece on π_i .

Figure 1 illustrates the “ragged honeycomb” structure of modified KV-puzzles. To formalize this, first note by inspection that the π_i do not intersect. Second we have:

Claim 3.4. *There is a bijective correspondence between the 1's on the λ -boundary and the \blacktriangle 's in π_1 . Specifically, the j th 1 on the λ -boundary is the terminus of the NWray of $\blacktriangle_{1,j}$. Similarly, there is a bijective correspondence between $\blacktriangle_{i+1,j}$ and $\mathbb{S}_{i,j}$ in that the former's NWray terminates at the southmost $/$ edge of the latter.*

Proof. Follows by combining Proposition 3.1 and Lemma 3.2. \square

Define \mathcal{L}_i to be the **left sequence** of π_i : Start at the southwest corner of $\Delta_{\lambda, \mu, \nu}$ and read the $\{\rightarrow, \nearrow\}$ -lattice path that starts along the ν -boundary and travels up the left boundary of π_i . The $\{0, 1\}$ -sequence records the labels of the edges seen. Similarly, define \mathcal{R}_i to be the **right sequence** of π_i by travelling up the right side of π_i but only reading the \rightarrow and \nearrow edges. (In Figure 1, $\mathcal{L}_1 = 0^6 10^{10} 1010 (= \lambda)$ while $\mathcal{R}_1 = 0^2 10^{11} 10^2 10^2$.)

In view of Claim 3.4, the following is “graphically” clear by considering the n diagonal strips through P :

Claim 3.5. $\mathcal{L}_1 = \lambda$, $\mathcal{L}_{i+1} = \mathcal{R}_i$ for $1 \leq i \leq k-1$, and $\mathcal{R}_k = \nu$.

Let $T^{(i)}$ be the tableau after adding labels of family $1, 2, \dots, i$. We declare $T^{(0)}$ to be the empty tableau of shape λ/λ . Let $\nu^{(i)}$ be the outer shape of $T^{(i)}$ (interpreted as the $\{0, 1\}$ -sequence for its lattice path).

Claim 3.6. $\mathcal{L}_i = \nu^{(i-1)}$ and $\mathcal{R}_i = \nu^{(i)}$.

Proof. Both assertions follow by inspection of the correspondences (B.1)–(B.9). (Also the second follows from the first, by Claim 3.5.) \square

It is straightforward from Claims 3.5 and 3.6 that $T = \phi(P)$ is semistandard in the sense of (S.1)–(S.4) of [PeYo15]. By Proposition 3.1, no label of T is \star -ed unless it is the rightmost

box label of its family in a row ($> i$). Since labels of family i are placed in the boxes of row i or below, no label of T can be too high. Since $\mathcal{R}_k = \nu$, the shape of T is ν/λ .

Claim 3.7. T has content μ .

Proof. Let β be the content of T . Then β_i is the number of (distinct) genes of family i that appear in T , which, in terms of P , is the number of \blacktriangleleft and \blacklozenge in π_i minus the number of purple KV-pieces \blacktriangledown in π_i . Thus the vertical height h_i of π_i (at its right endpoint) is $\beta_i + \# \blacktriangle$. However, h_i equals the number of line segments strictly below the i th 1 on the μ -boundary; i.e., $h_i = n - i - (n - k - \mu_i) = (k - i) + \mu_i$. By Claims 3.4 and 3.1, $\# \blacktriangle = (k - i)$, hence $\beta = \mu$, as desired. \square

Finally,

Claim 3.8. T is ballot.

Proof. The **height** of a (combination) piece is the distance of any northernmost point to the ν -boundary as measured along any (anti)diagonal. The height h of $\blacktriangle_{i+1,j}$ equals the number of \blacktriangleleft 's, \blacktriangle 's and \blacklozenge 's that appear weakly before $\blacktriangle_{i+1,j}$ in π_{i+1} minus the number of \blacktriangledown 's before $\blacktriangle_{i+1,j}$ in π_{i+1} . There are exactly j such \blacktriangle 's, while the number of \blacktriangleleft 's and \blacklozenge 's is the number of labels used and the number of \blacktriangledown 's is the number of these labels that are repeats. That is $h = j + (\# \text{distinct genes of family } i+1 \text{ in row } j+1 \text{ and above})$ where we do *not* include labels on the lower edges of row $j+1$. Similarly, the height h' of $\mathbb{S}_{i,j}$ is given by $h' = j + (\# \text{distinct genes of family } i \text{ in row } j \text{ and above})$ where we include labels on the lower edges of row j . By Claim 3.4, $h' - h \geq 0$ and so ballotness follows. \square

3.3. Well-definedness of $\psi : \mathcal{T} \rightarrow \mathcal{P}$. Let $T \in \mathcal{T}$ be a starred ballot genomic tableau of shape ν/λ and content μ . Let $P = \psi(T)$. Let π_i be the track associated to σ_i . As in Section 3.2, we define the $\{0, 1\}$ -sequences \mathcal{L}_i and \mathcal{R}_i associated to π_i . Here, $T^{(i)}$ is defined as the subtableau of T using the labels of family $1, 2, \dots, i$. Hence $T^{(0)}$ is the empty tableau of shape λ/λ . Let $\nu^{(i)}$ be the outer shape of $T^{(i)}$.

Claim 3.9 (cf. Claim 3.6). $\mathcal{L}_i = \nu^{(i-1)}$ and $\mathcal{R}_i = \nu^{(i)}$.

Proof. By inspection of the correspondences (B.1)-(B.9). \square

By the lattice path definition, each $\nu^{(j)}$ is a length n sequence. So π_i is a track that (by definition) starts at the south border of Δ and terminates at the east border of Δ . Also, define $\blacktriangle_{i,j}$ and $\mathbb{S}_{i,j}$ as before.

Claim 3.10. $\mathbb{S}_{i,j}$ and $\blacktriangle_{i+1,j}$ share a diagonal with the former strictly northwest of the latter.

Proof. The 1's in \mathcal{L}_{i+1} result solely from the \blacktriangle 's appearing in π_{i+1} while the 1's appearing in \mathcal{R}_i result solely from the \mathbb{S} (combination) pieces. Thus, that the pieces share a diagonal follows from Claim 3.9. For the "northwest" assertion, repeat Claim 3.8's argument but reverse the logic of the final sentence: since by assumption T is ballot, it follows that $h' \geq h$. \square

Since Claims 3.9 and 3.10 combine to imply that the π_i are non-intersecting, attaching NWrays to each \blacktriangle and filling with white 0-triangles as prescribed, we have a filling P of $\Delta_{\tilde{\lambda}, \tilde{\mu}, \nu}$ satisfying the modified KV-puzzle rule. It remains to check the λ - and μ -boundaries.

Claim 3.11. $\tilde{\lambda} = \lambda$.

Proof. Graphically, $\tilde{\lambda} = \mathcal{L}_1$. On the other hand, by Claim 3.9, we know that $\mathcal{L}_1 = \lambda$. \square

Claim 3.12. $\tilde{\mu} = \mu$.

Proof. This is given by reversing the logic of the proof of Claim 3.7; here we are given the content of T and are determining the heights of the tracks π_i . \square

3.4. Weight-preservation. We wish to show:

Claim 3.13. ϕ is weight-preserving, i.e., $\text{wt}(P) = \widehat{\text{wt}}(T)$.

Proof. The ± 1 sign associated to P and T is the same since each usage of a KV-piece in P corresponds to a \star -ed label or a repetition of a gene in T .

Now consider the weight $1 - \frac{t_a}{t_b}$ assigned to an equivariant piece p in P . Here a is the ordinal (counted from the right) of the line segment s on the ν -boundary hit by the diagonal “right leg” emanating from p . Then b equals $a + h - 1$ where h is the height of the piece p . Suppose p lies in track π_i , and corresponds either to i_j on the lower edge of box x in row r or to $i_j^* \in x$ in row r . Consider the edge e on the left boundary of π_i that is on the same diagonal as s . If p is not attached to the first KV-piece, so it corresponds to an edge label, then e ’s index from the right in the string \mathcal{L}_i equals $\text{Man}(x)$. Otherwise e ’s index from the right in the string \mathcal{L}_i equals $\text{Man}(x) + 1$.

Note that h equals the number of \blacktriangleleft ’s, \blacktriangle ’s and \blacklozenge ’s appearing weakly before p in π_i minus the number of \blacktriangledown ’s appearing before p in π_i . The number of such \blacktriangle ’s equals $1 + r - i$ if p corresponds to an edge label and equals $r - i$ if p corresponds to a starred label. The number of such \blacktriangleleft ’s and \blacklozenge ’s minus the number of such \blacktriangledown ’s equals $\mu_i - j + 1$. Weight preservation follows. \square

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